Mass and Spin of Kerr black holes from observations.

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November 2019

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Brief general motivation

- It exists dynamical evidence of the existence of a supermassive black hole in the center of the milky way (named SgrA*).
  M. B. Begelman,Science **300**, 1898 (2003); Z. Q. Shen et al., Nature (London) **438**, 62 (2005); A. M. Ghez et al., Astrophys. J. **689**, 1044 (2008).
- In General Relativity theory, in four dimensions and vacuum, neutral black holes are described completely by the family of Kerr metrics and characterized by two physical parameters: **the mass M and the angular momentum parameter** $a = J/M$, where J is the angular momentum of the rotating black hole.
- For SgrA*, $M \sim 3.6 \times 10^6 M_\odot$.
- **Goal of this talk: To obtain the parameters of compact objects (M and a for the Kerr black hole) in terms of the blueshifts and redshifts of photons emitted by massive particles or gas moving in geodesic orbits around a Kerr black hole.**

Metric and Killing vectorial fields

We choose the most general axisymmetric stationary metric with two orthogonal planes in 3+1 dimensions: (we choose spherical coordinates $x^\mu(t, r, \theta, \varphi)$ and the gauge $g_{r\theta} = 0$)

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \tag{1}$$

With the following functional dependence,

$$g_{\mu\nu}(r, \theta) \qquad \text{for} \qquad \mu, \nu = t, r, \theta, \varphi \tag{2}$$

The metric (1) has two commuting Killing vectorial fields: $[\xi, \psi] = 0$:

$$\xi^\mu = (1, 0, 0, 0) \qquad \text{Killing stationary timelike vectorial field} \tag{3}$$

$$\psi^\mu = (0, 0, 0, 1) \qquad \text{Killing rotational vectorial field} \tag{4}$$

Geodesics and their conserved quantities. Carter, Phys. Rev. 174, 1559 (1968)

The emitter of photons is a massive particle following a geodesic curve on the spacetime with metric (1) and velocity:

$$U_e^\mu = (U^t, U^r, U^\theta, U^\varphi)_e \tag{5}$$

Due to the existence of the Killing vector fields (3)-(4), it exist conserved quantities along of the geodesic of the massive particle: The total Energy E and the component of the total angular momentum L around the symmetry zeta axe (both per unit of mass):

$$E = \frac{\bar{E}}{m} = -g_{\mu\nu}\xi^\mu U^\nu = -g_{tt}U^t - g_{t\varphi}U^\varphi \tag{6}$$

$$L = \frac{\bar{L}}{m} = g_{\mu\nu}\psi^\mu U^\nu = g_{\varphi t}U^t + g_{\varphi\varphi}U^\varphi \tag{7}$$

Geodesics and conserved quantities

From (7) y (8), we obtain U^t y U^φ in terms of E y L,

$$U^t = \frac{(Eg_{\varphi\varphi} + Lg_{t\varphi})}{(g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})} \tag{8}$$

$$U^\varphi = -\frac{(Eg_{t\varphi} + Lg_{tt})}{(g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})} \tag{9}$$

The normalization of the velocity of the particle implies the equation:

$$-1 = U^\mu U_\mu = g_{tt}(U^t)^2 + 2g_{t\varphi}U^tU^\varphi + g_{\varphi\varphi}(U^\varphi)^2 + g_{rr}(U^r)^2 + g_{\theta\theta}(U^\theta)^2$$

Introducing the velocities U^t y U^φ in the previous equation:

$$g_{rr}(U^r)^2 + g_{\theta\theta}(U^\theta)^2 = \frac{E^2 g_{\varphi\varphi} + 2E \cdot Lg_{t\varphi} + L^2 g_{tt}}{(g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi})} - 1 \tag{10}$$

Family of Kerr black holes in Boyer-Lindquist coordinates: $M^2 \geq a^2$

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

with the metric components,

$$g_{tt} = -\left[1 - \frac{2Mr}{\Sigma}\right], \quad g_{t\varphi} = -\left[\frac{2Mar\sin^2\theta}{\Sigma}\right], \quad g_{rr} = \frac{\Sigma}{\Delta},$$

$$g_{\varphi\varphi} = \left[r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right]\sin^2\theta\,, \quad g_{\theta\theta} = \Sigma\,.$$

where we have:

$$\Delta = r^2 + a^2 - 2Mr\,, \qquad \Sigma = r^2 + a^2\cos^2\theta\,,$$

$$g_{t\varphi}^2 - g_{\varphi\varphi}g_{tt} = \Delta\sin^2\theta\,, \qquad M^2 \geq a^2\,.$$

The Killing tensor in the metric Kerr and the Carter constant.

The Kerr metric has a Killing tensor field $K_{\mu\nu}$:

$$K_{\mu\nu} = 2\Sigma\, l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu} \quad \text{satisface} \quad \nabla_{(\alpha} K_{\mu\nu)} = 0\,,$$

where we have the null vectorial fields (l^{μ}, n^{μ}) satisfying $l^{\mu} l_{\mu} = n^{\mu} n_{\mu} = 0$ and $l^{\mu} n_{\mu} = -1$,

$$\begin{aligned} l^{\mu} &= \frac{r^2 + a^2}{\Delta}\left(\frac{\partial}{\partial t}\right)^{\mu} + \frac{a}{\Delta}\left(\frac{\partial}{\partial \varphi}\right)^{\mu} + \left(\frac{\partial}{\partial r}\right)^{\mu}\,, \\ n^{\mu} &= \frac{r^2 + a^2}{2\Sigma}\left(\frac{\partial}{\partial t}\right)^{\mu} + \frac{a}{2\Sigma}\left(\frac{\partial}{\partial \varphi}\right)^{\mu} - \frac{\Delta}{2\Sigma}\left(\frac{\partial}{\partial r}\right)^{\mu}\,, \end{aligned}$$

It is known the existence of a motion constant C:

$$\Rightarrow \quad C = K_{\mu\nu} U^{\mu} U^{\nu} = 2\Sigma\,(l_{\mu} U^{\mu})(n_{\mu} U^{\mu}) - r^2 = \text{Constant}$$

Equations of geodesic motion of particles

The constant C is written in terms of the Carter constant Q:

$$C \equiv (L - aE)^2 + Q = \left(\left[(r^2 + a^2)E - aL \right]^2 - \Sigma^2 (U^r)^2 - \Delta r^2 \right) \left(\frac{1}{\Delta} \right) \quad ,$$

from which we obtain the radial velocity component U^r:

$$\Sigma^2 (U^r)^2 = \left[(r^2 + a^2)E - aL \right]^2 - \Delta \left[r^2 + (L - aE)^2 + Q \right] \equiv V^2(r)$$

Using the previous equation in (10) we have for U^θ:

$$\Sigma^2 (U^\theta)^2 = Q - \left[a^2 (1 - E^2) + \frac{L^2}{\sin^2\theta} \right] \cos^2\theta \equiv \Theta^2(\theta)$$

The physical interpretation of the constant Carter Q is obtained from the last equation:

Equations of geodesic motion.

The Carter constant Q satisfies:

$$Q = \Sigma^2 (U^\theta)^2 + \left[a^2 (1 - E^2) + \frac{L^2}{\sin^2\theta} \right] \cos^2\theta$$

For bound orbits (orbits not reaching $r \to \infty$) have:

- $E < 1$.
- $Q \geq 0$
- $Q = 0 \quad \text{for} \quad \theta = \pi/2 \quad \Leftrightarrow \quad \text{equatorial orbits}$

For unbound orbits (orbits reaching $r \to \infty$) have:

$$E \geq 1$$

Geodesics equations for U^μ with given parameters (E, L, Q, x_o^μ)

$$U^t = \frac{1}{\Delta\Sigma}\left\{\left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right] E - (2Mar)\, L\right\}$$

$$U^\varphi = \frac{1}{\Delta\Sigma \sin^2\theta}\left[(2Mar\sin^2\theta)\, E + (\Delta - a^2\sin^2\theta)\, L\right]$$

$$\Sigma^2 (U^r)^2 = \left[(r^2+a^2)E - aL\right]^2 - \Delta\left[r^2 + (L-aE)^2 + Q\right] \equiv V^2(r)$$

$$\Sigma^2 (U^\theta)^2 = Q - \left[a^2(1-E^2) + \frac{L^2}{\sin^2\theta}\right]\cos^2\theta \equiv \Theta^2(\theta)$$

Detection and emission of photons with momentum k^μ.

A Photon with momentum k^μ is given by:

$$k^\mu = (k^t, k^r, k^\theta, k^\varphi) \quad ,$$

The photon is moving on a null geodesic outside of the event horizon of the Kerr black hole,

$$0 = k^\mu k_\mu = g_{tt}(k^t)^2 + 2g_{t\varphi}(k^t k^\varphi) + g_{\varphi\varphi}(k^\varphi)^2 + g_{rr}(k^r)^2 + g_{\theta\theta}(k^\theta)^2 \quad ,$$

and it has the motion constants (Q_γ is the corresponding Carter constant):

$$\begin{aligned} E_\gamma &= -g_{\mu\nu}\xi^\mu k^\nu = -g_{tt}k^t - g_{t\varphi}k^\varphi \quad , \\ L_\gamma &= \quad g_{\mu\nu}\psi^\mu k^\nu = g_{\varphi t}k^t + g_{\varphi\varphi}k^\varphi \quad , \end{aligned}$$

$$C_\gamma \equiv (L_\gamma - aE_\gamma)^2 + Q_\gamma = K_{\mu\nu}k^\mu k^\nu = 2\Sigma\,(l_\mu k^\mu)(n_\mu k^\mu)$$

Geodesic equation for k^μ with given parameters $(E_\gamma,\ L_\gamma,\ Q_\gamma,\ y_o^\mu)$

$$k^t = \frac{1}{\Delta\Sigma}\left\{\left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right] E_\gamma - (2Mar)\, L_\gamma\right\}$$

$$k^\varphi = \frac{1}{\Delta\Sigma\sin^2\theta}\left[(2Mar\sin^2\theta)\, E_\gamma + (\Delta - a^2\sin^2\theta)\, L_\gamma\right]$$

$$\Sigma^2 (k^r)^2 = \left[(r^2+a^2)E_\gamma - aL_\gamma\right]^2 - \Delta\left[(L_\gamma - aE_\gamma)^2 + Q_\gamma\right]$$

$$\Sigma^2 (k^\theta)^2 = Q_\gamma - \left[-a^2 E_\gamma^2 + \frac{L_\gamma^2}{\sin^2\theta}\right]\cos^2\theta$$

Redshifts and blueshifts of photons emitted by massive particles.

The frequency of a photon measured by a particle with velocity U^{μ}:

$$\omega_e = -(k_{\mu} U^{\mu}) \mid_e$$

The detected frequency by an observer far away at infinity ($r \to \infty$) with velocity $U^{\mu} \mid_d$ is:

$$\omega_d = -(k_{\mu} U^{\mu}) \mid_d$$

where the velocities of the emitter (e) and the detector (d) are:

$$U_e^{\mu} = (U^t, U^r, U^{\theta}, U^{\varphi}) \mid_e$$

$$U_d^{\mu} = (U^t, 0, 0, U^{\varphi}) \mid_d$$

Redshifts and blueshifts emitted by photons.

The observed shifts (red and blue) are defined by:

$$1 + z = \frac{\omega_e}{\omega_d} = \frac{(E_\gamma U^t - L_\gamma U^\varphi - g_{rr} U^r k^r - g_{\theta\theta} U^\theta k^\theta)\,|_e}{(E_\gamma U^t - L_\gamma U^\varphi)\,|_d}$$

In general, we have a function F:

$$1 + z = \frac{\omega_e}{\omega_d} = F(r, \theta, b, B, q, s)$$

where the parameters (b, B, q, s) are defined by:

$$b \equiv \frac{L_\gamma}{E_\gamma} \quad , \quad B \equiv \frac{L}{E} \quad , \quad q \equiv \frac{Q_\gamma}{E_\gamma} \quad , \quad s \equiv \frac{Q}{E}$$

Shifts for photons emitted by particles in circular orbits.

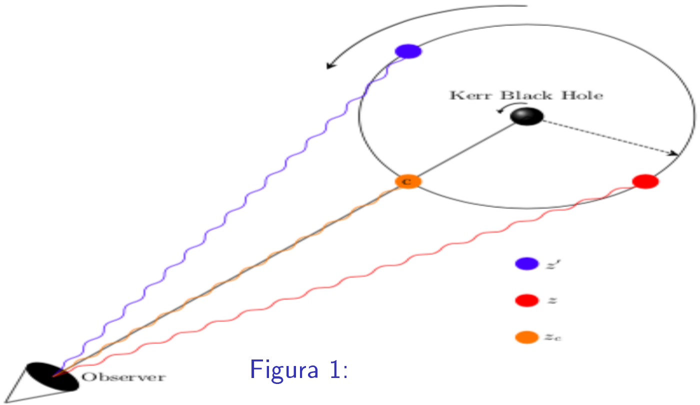

Figura 1:

Toy model: equatorial circular orbits ($\theta = \pi/2$).

For equatorial circular orbits we have $U^r = U^\theta = 0$ (remember that $b = L_\gamma/E_\gamma$) and therefore:

$$1 + z = \frac{(E_\gamma U^t - L_\gamma U^\varphi)\,|_e}{(E_\gamma U^t - L_\gamma U^\varphi)\,|_d} = \frac{(U^t - b\,U^\varphi)\,|_e}{(U^t - b\,U^\varphi)\,|_d}$$

Where the detector is sufficiently far away ($r \to \infty$), we have:

$$U^\mu_d = (U^t, 0, 0, U^\varphi)\,|_d \qquad \to \qquad (1, 0, 0, 0)$$

This is the limit that we study in this talk.

$$1 + z = (U^t - b\,U^\varphi)\,|_e$$

E y L for equatorial circular orbits.

U^φ, U^t, E y L for equatorial circular orbits are:

$$U^\varphi(r,\pi/2) = \frac{(2Ma)E + (r-2M)L}{r(r^2+a^2-2Mr)}$$

$$U^t(r,\pi/2) = \frac{(r^3+a^2r+2Ma^2)E - (2Ma)L}{r(r^2+a^2-2Mr)}$$

$$E = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4}\left(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}\right)^{1/2}}$$

$$L = (\pm)\frac{M^{1/2}\left(r^2 \mp 2aM^{1/2}\, r^{1/2} + a^2\right)}{r^{3/4}\left(r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}\right)^{1/2}}$$

Shifts for photons emitted by particles in circular orbits.

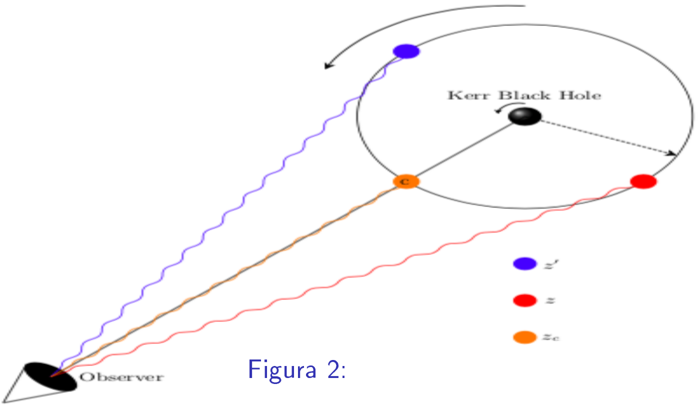

Figura 2:

The parameter b for photons emitted in both arms of the circular orbit.

We choose the value of the impact parameter $b = L_\gamma / E_\gamma$ as the value for which $k_e^r = 0$, corresponding to the points on the horizontal axe perpendicular to the null geodesic in the emission point of the photon. Using the null property of photons $k^\mu k_\mu|_e = 0$, we find:

$$b_\pm = \frac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt} g_{\varphi\varphi}}}{g_{tt}}$$

Using the Kerr metric, we find ($k_e^r = 0$):

$$b_\pm = \frac{(2Ma) \mp r\sqrt{r^2 + a^2 - 2Mr}}{r - 2M}$$

The parameter b for photons emitted in front of signal line.

In this case the value of the impact parameter $b_c = L_\gamma / E_\gamma$ is the value for which $k_e^\varphi = 0$, corresponding to the points on the axe parallel to the null geodesic in the emission point of the photon. We find:

$$b_c = \frac{-g_{t\varphi}}{g_{tt}}$$

Using the Kerr metric, we find ($k_e^\varphi = 0$):

$$b_c = \frac{(2Ma)}{r - 2M}$$

Structure of the shifts of the Schwarzschild black hole

$$
\begin{aligned}
Z &= Z_c + Z_{kin}, \\
Z_c &= \frac{1}{\sqrt{1 - 3\left(\frac{M}{r}\right)}} - 1 \\
Z_{kin} &= (\pm)\frac{1}{\sqrt{1 - 3\left(\frac{M}{r}\right)}}\frac{1}{\sqrt{1 - \frac{2M}{r}}}\left(\frac{M}{r}\right)^{1/2}
\end{aligned}
$$

Signs between curve parenthesis mean kinematic red and blue shifts.
Term in red color is the shift due to NEWTONIAN CIRCULAR VELOCITY.
Term in blue color is the GRAVITATIONAL RED SHIFT.

Structure of the shifts of the Kerr black hole

$$
\begin{aligned}
Z &= Z_c \pm Z_{kin}, \\
1 + Z_c &= \frac{1}{\sqrt{1 - 3\left(\frac{M}{r}\right) \pm 2\left(\frac{a}{r}\right)\left(\frac{M}{r}\right)^{1/2}}} \left[1 \pm \left(\frac{a}{r}\right)\left(\frac{M}{r}\right)^{1/2} \pm 2\left(\frac{a}{r}\right) \frac{\left(\frac{M}{r}\right)^{3/2}}{\left(1 - \frac{2M}{r}\right)} \right] \\
Z_{kin} &= (\pm) \frac{1}{\sqrt{1 - 3\left(\frac{M}{r}\right) \pm 2\left(\frac{a}{r}\right)\left(\frac{M}{r}\right)^{1/2}}} \left(\frac{M}{r}\right)^{1/2} \left[\frac{\sqrt{1 - \frac{2M}{r} + \left(\frac{a}{r}\right)^2}}{\left(1 - \frac{2M}{r}\right)} \right]
\end{aligned}
$$

Superior signs corresponding to co-rotating orbits with the rotation of the black hole
Inferior signs to counter-rotating orbits with the rotation of the black hole.
Signs between curve parenthesis mean kinematic red and blue shifts.
Term in red color is the shift due to NEWTONIAN CIRCULAR VELOCITY.
Term in blue color is the GRAVITATIONAL RED SHIFT.

Shifts of photons emitted by masers in NGC 4258.Moran et al.

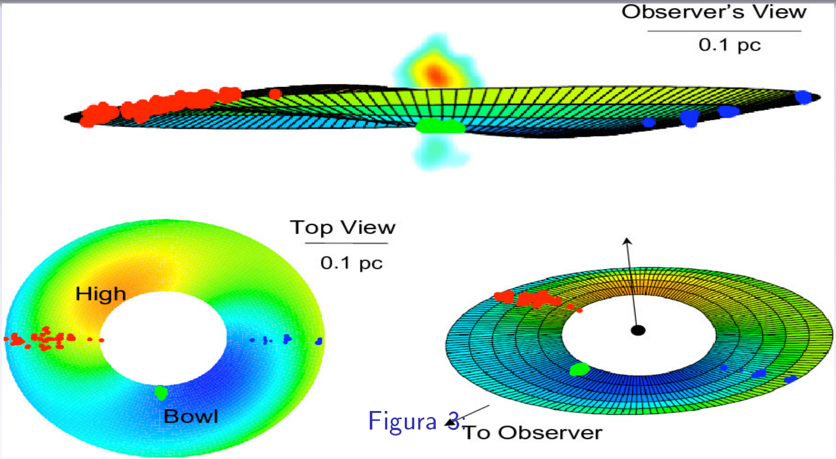

Figura 3:

Shifts of photons by masers in NGC 4258. Moran et al.

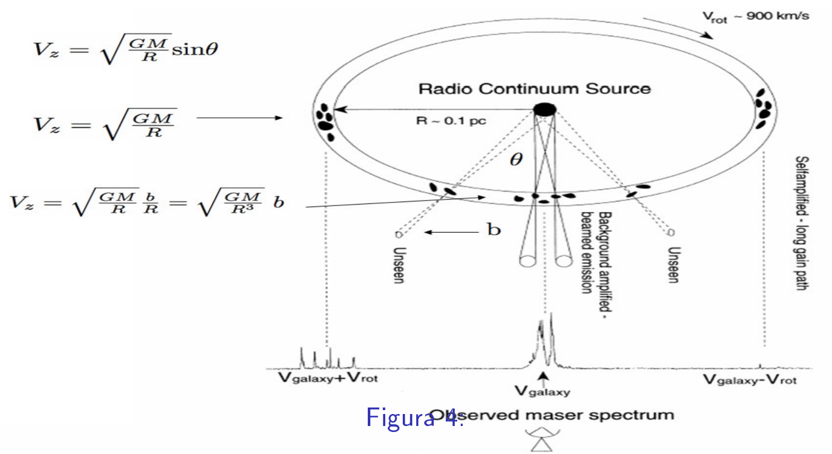

Figura 4:

The accretion disk of NGC 4258.

Accretion disk material emits H_2O maser photons emission at 22.235 GHz with high velocities Doppler components:

- $V \approx V_{sys} \pm 1000\ Km/sec$.
- Systematic Velocity: $V_{sys} = 472 \pm 0.4\ Km/sec$
- Distance form Milky Way: $7.2 \pm 0.3\,(random) \pm 0.4\,(systematic)$ Mpc
- Bayesian Estimation for the Mass: $M = 3.73 \pm 0.0014 \times 10^7 M_\odot$
- There is not sensibility for the rotation parameter:

$$0 \leq \frac{a}{M} \leq 1$$

Conclusions

- Simple Method to determine the parameters M y a of a Kerr black hole in terms of the cinematical red and blue shifts of photons emitted by massive particle moving in geodesics on the Kerr black hole (*Rotation Curves of the Kerr black holes*).
- Future work: To compute the shifts for general orbits: non equatorial circular, equatorial eliptic orbits, eliptic no equatorial orbits, etc.
- To use simulated data samples in order to estimate the parameters (M, a) doing a Bayesian estimation of parameters. This analysis will permit to estimate the precision of the shifts to be measured.

References

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