La Noción de Cuantización en Matemáticas.

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- R-vector and Multivectors
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- The Clifford Algebra
- Maxwell and Hamilton Equations
- New construction of Clifford Algebra
- Relation of Clifford Algebra an the Exterior Algebra
- Quantization in mathematics
- Geometric Quantization and Noncommutative geometry
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Motivations

The main motivation for this short talk is to give a survey on the mathematical treatment of quantization and the notion mathematicians have about this topic. Of course, this subject comes from physics where quantization has a physical meaning, it's the change in behavior particles have when we study subatomic scales and how the theory has to being adequated to fit the experimental results.

Algebra

Remember the definition of an algebra

Definition

A vector space V equipped with a product is called an **Algebra**.

r-vectors

The usual geometric interpretation of vectors as arrows give us the intuition to abstract and generalize this notion. So, a vector space V of $dim V = 1$ contain at most 1-vectors, one of $dim V = 2$ contain 2-vectors but also 1-vectors. With this in mind, we can regard scalars as 0-vectors and interpret them as points, 1-vectors as directed arrows and 2-vectors as directed planes. Then 3-vectors are directed volumes and an **r-vector** is a directed hypersurface. Let us write r-vectors as A_r and call it of **grade** r.

Multivectors

Now we can define

Definition

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$A = A_0 + A_1 + A_2 + \cdots + A_r$

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This new object result to have interesting features. At first it could seem ill defined, since is the sum of objects of different dimension. The notion relies in the concept of r-vector which, in fact, is not new; directed surfaces are classically constructed as the result of operation between usual vectors, here we defined them as pure objects contained on a space of greater dimension. Additionally, since sum between r-vector is well defined, also is the sum with multivectors. Do not forget that r-vectors are still elements of the vector space V. Actually, we can prove that multivectors form a vector space.

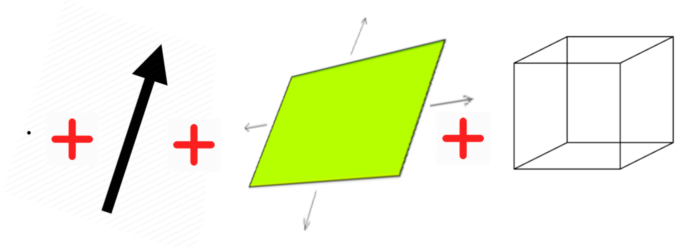

Figure: Geometric interpretation of a multivector

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Definition

$A_r \cdot B_r := \langle A_r B_r \rangle_{|r-s|}$ *Inner product*

$A_r \wedge B_r := \langle A_r B_r \rangle_{|r+s|}$ *Exterior product*

In this sense, inner product is an operation that decrease the grade of a r-vector and exterior product increase it. Note that this definitions recovers the usual sense when dealing with ordinary vectors (1-vectors).

For basis vectors

$$e_i e_i = {e^2}_i = 1$$

In order to calculate $e_i e_j$ for $i \neq j$ let us do

$$\begin{aligned}
&(e_i + e_j)(e_i + e_j) = e_i e_i + e_i e_j + e_j e_i + e_j e_j \\
&(e_i + e_j)^2 = 2 + e_i e_j + e_j e_i \\
&(e_i + e_j) \cdot (e_i + e_j) = 2 + e_i e_j + e_j e_i \\
&e_i \cdot e_i + e_i \cdot e_j + e_j \cdot e_i + e_j \cdot e_j = 2 + e_i e_j + e_j e_i \\
&2 = 2 + e_i e_j + e_j e_i
\end{aligned}$$

$$\Rightarrow e_i e_j = -e_j e_i \tag{1}$$

Now, to facilitate calculus, we express the geometric product of two vectors in $\mathcal{C}(V)$ with $\dim V = 2$.

$$
\begin{aligned}
a &= a_1 e_1 + a_2 e_2 \\
b &= b_1 e_1 + b_2 e_2 \\
ab &= (a_1 e_1 + a_2 e_2)(b_1 e_1 + b_2 e_2) \\
ab &= a_1 b_1 + a_2 b_2 e_2 e_2 + a_1 b_2 e_1 e_2 + a_2 b_1 e_2 e_1 \\
ab &= a_1 b_1 + a_2 b_2 + (a_1 b_2 - a_2 b_1) e_1 e_2 \\
ba &= (b_1 e_1 + b_2 e_2)(a_1 e_1 + a_2 e_2) \\
ba &= a_1 b_1 + a_2 b_2 + (b_1 a_2 - b_2 a_1) e_1 e_2
\end{aligned}
$$

And then,

$$
\begin{aligned}
&ab + ba = 2a_1b_1 + 2a_2b_2 = 2(a_1b_1 + a_2b_2) \\
&\Rightarrow \frac{1}{2}(ab + ba) = a_1b_1 + a_2b_2 = \langle ab \rangle_0 = a \cdot b \\
&ab - ba = 2a_1b_2e_1e_2 - 2a_2b_1e_1e_2 = 2(a_1b_2 - a_2b_1)e_1e_2 \\
&\Rightarrow \frac{1}{2}(ab - ba) = (a_1b_2 - a_2b_1)e_1e_2 = \langle ab \rangle_2 = a \wedge b
\end{aligned}
$$

This is

$$a \cdot b = \frac{1}{2}(ab + ba) \tag{2}$$

$$a \wedge b = \frac{1}{2}(ab - ba) \tag{3}$$

By sum of (2) and (3), we obtain

$$ab = a \cdot b + a \wedge b \tag{4}$$

This result express the geometric product of two vectors.

All the identities of Gibbs vector algebra are contained here. Additionally, note this interesting feature.

$$
\begin{aligned}
(e_1 e_2)(e_1 e_2) &= (e_1 e_2)(-e_2 e_1) \\
(e_1 e_2)^2 &= -(e_1 e_2)(e_2 e_1) \\
(e_1 e_2)^2 &= -e_1(e_2 e_2)e_1 \\
(e_1 e_2)^2 &= -e_1 e_1 \\
(e_1 e_2)^2 &= -1
\end{aligned}
$$

The bivector $e_1 e_2$ posses the property that its squares is -1, hence it is identical to the imaginary unity i. It results that geometric product naturally arises the structure of complex spaces over vectors. To make it clear, let us examine more the Clifford Algebra over vector space of dimension 2 ($\mathcal{C}_2$). Here, the multivectors are

$$A = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_{12}$$

With $a_i \in \mathbb{R}$ and adopting $e_{12} := e_1 e_2$.

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With $a_i \in \mathbb{R}$ and adopting $e_{12} := e_1 e_2$. Note that, since the basis of V is $\{e_1, e_2\}$ the dimension of $\mathcal{C}_2(V)$ is $2^2 = 4$ and its basis is $\{1, e_1, e_2, e_{12}\}$. A multiplication table is shown below.

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

Elements of $\mathcal{C}_2$ are sum of elements of $\mathbb{R}$, $\mathbb{R}^2$, in fact, it is the direct sum.

$$\mathcal{C}_2 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2$$

As well, note that every multivector could be reordered and be expressed as the sum of its even parts and its odd parts

$$A = A_0 + A_1 + A_2 + \cdots + A_n = (A_0 + A_2 + \cdots A_n) + (A_1 + A_3 + \cdots A_{n-1})$$

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In the specific case of $\mathcal{C}_2$ we have

$$\mathcal{G}_2^{(+)} = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$$
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So, elements of $\mathcal{C}_2^{(+)}$ are of the form $a_1 + a_2 e_{12}$. Thus, $\mathcal{C}_2^{(+)} \simeq \mathbb{C}$, *i.e.*, they are isomorphic. Having $\mathbb{C}$ as a subalgebra of $\mathcal{C}_2$ make us realize the powerful of the geometric product. If we look for GA over vector space of dimension 1 we, in fact, get $\mathbb{C}$ for which the even subalgebra is $\mathbb{R}$. This natural expression of the language of algebra and vectors makes GA a suitable theory to write physics. Actually, Clifford Algebra over V of dimension 4 results in what is called *Space Time Algebra* or *Dirac Algebra*, which even subalgebra is the *Pauli Algebra*, and which correspondent even subalgebra is $\mathbb{H}$ (quaternion space).

As the differential operator. Considering

$$\partial \cdot F = e_i \cdot \partial_i F \qquad \partial \wedge F = e_i \wedge \partial_i F$$

In similar way made previously, for vector valued function we end with

$$\partial F = \partial F + \partial \wedge F$$

This is an expression for the derivative of F that allow us to write in very beautiful way some equations.

For example, define the multivectors in $\mathcal{C}_4$

$$F = E + iB \qquad \mathcal{J} = \rho + J$$

F is a bivector of the electromagnetic field and $\mathcal{J}$ is the sum of the charge density ρ and the current density J. Setting this, we can write Maxwell Equations as

Maxwell Equations

$\partial F = \mathcal{J}$

This is

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Now, express F in its basis. For brevity let us adopt the Einstein summation convention is this calculus

$$\partial \cdot F = \gamma^{\mu}\partial_{\mu} \cdot F^{\mu\nu}\gamma_{\mu}\gamma_{\nu} = \boldsymbol{\partial_{\mu}\mathbf{F}^{\mu\nu}\gamma_{\nu} = \mathbf{J}^{\nu}\gamma_{\nu}}$$
$$\partial \wedge F = i(\partial \times F) = \partial \times iF = \gamma^{\mu}\partial_{\mu}F_{\mu\nu}\gamma_{\mu}\gamma_{\nu} = \boldsymbol{\partial_{\mu}\mathbf{F}_{\mu\nu}\gamma_{\nu} = 0}$$

We recognize this as the Maxwell Equation in tensorial form.

Now let us explore the Cl.A. associated to the vector space $\mathbb{R}^n_{(q)} \oplus \mathbb{R}^n_{(p)}$. This is with the intention to set a basis $\{\hat{q}_i\}_{i=1}^n$ that span $\mathbb{R}^n_{(q)}$ and $\{\hat{p}_i\}_{i=1}^n$ that span $\mathbb{R}^n_{(p)}$. We regard $q = \sum_{i=1}^{n} q_i \hat{q}_i$ as the *position* and $p = \sum_{i=1}^{n} p_i \hat{p}_i$ as the *momentum*.

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$$\omega = \sum_{k=1}^{n} \omega_k = \sum_{k=1}^{n} \hat{q}_k \wedge \hat{p}_k = \hat{q} \wedge \hat{p}$$

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This induce a map

$$\tilde{\omega} \colon \mathbb{R}^n_{(q)} \oplus \mathbb{R}^n_{(p)} \to \mathbb{R}$$
$$x \mapsto \tilde{x} = x \cdot \omega$$

$$\tilde{x} = x \cdot \omega = x \cdot (\hat{q} \wedge \hat{p}) = (x \cdot \hat{q})\hat{p} - (x \cdot \hat{p})\hat{q}$$

But $x = q\hat{q} + p\hat{p}$

$$\tilde{x} = ((q\hat{q} + p\hat{p}) \cdot \hat{q})\hat{p} - ((q\hat{q} + p\hat{p}) \cdot \hat{p})\hat{q}$$
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$$\begin{aligned} \tilde{x} &= ((q\hat{q} + p\hat{p}) \cdot \hat{q})\hat{p} - ((q\hat{q} + p\hat{p}) \cdot \hat{p})\hat{q} \\ \tilde{x} &= q\hat{p} - p\hat{q} \end{aligned}$$

If we do this with the derivative operator we obtain

$$\tilde{\omega}(\partial) = \tilde{\partial} = \hat{p}\partial_q - \hat{q}\partial_p$$

And express Hamilton equations as

Hamilton equations

$$\tilde{\partial} H = \dot{x}$$

With a quick calculus we return to classical expression

$$(\hat{p}\partial_q - \hat{q}\partial_p)H = \dot{p}\hat{p} + \dot{q}\hat{q}$$
$$\partial_q H = \dot{p} \qquad - \partial_p H = \dot{q}$$

Geometric Aspects

Recall that endow a space with a bilinear form allow us to give geometrical sense such as distance, lenght, angles, etc. For example, $\mathbb{R}^2$ endowed with a symmetric and positive definite bilinear form is the Euclidean Plane.

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Recall that endow a space with a bilinear form allow us to give geometrical sense such as distance, lenght, angles, etc. For example, $\mathbb{R}^2$ endowed with a symmetric and positive definite bilinear form is the Euclidean Plane. Now, consider two vector spaces V and W. Define a bilinear function by

$$\begin{aligned} B\colon V \times W &\to \mathbb{K} \\ (x, y) &\mapsto B(x, y) \end{aligned}$$

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$$\begin{aligned} B\colon V \times W &\to \mathbb{K} \\ (x, y) &\mapsto B(x, y) \end{aligned}$$

Associated, we have

$$\begin{aligned} B_x\colon V &\to \mathbb{K} \\ x &\mapsto B(x, y) \end{aligned}$$

$$\begin{aligned} B_y\colon W &\to \mathbb{K} \\ y &\mapsto B(x, y) \end{aligned}$$

Clearly, $B_x \in V*$ and $B_y \in W*$.

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$$\gamma \colon V \to W* \\ x \mapsto B_y$$

function

$$\delta \colon W \to V* \\ y \mapsto B_x$$

So, $\ker\gamma \equiv W' \subset V$ and $\ker\delta \equiv V' \subset W$. V' and W' are called the **conjugated** of V and W, respectively. Given this, we say that B is **non-degenerate** if $W' = V' = \{0\}$

Now, let's assume $V = W$. The choices of B which lead to $V' = W'$ are the same to take $B(x, y) = 0$ equivalent with $B(y, x) = 0$ for all $(x, y) \in V \times V$. This holds whenever B is neither symmetric or antisymmetric.

Proposition

If $\forall (x, y) \in V \times V$, $B(x, y) = 0 \Leftrightarrow B(y, x) = 0$, then B must be symmetric or antisymmetric.

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Proposition

If $\forall (x, y) \in V \times V$, $B(x, y) = 0 \Leftrightarrow B(y, x) = 0$, then B must be symmetric or antisymmetric.

You can find a proof of this proposition in [2].
If B is symmetric the resulting geometry is called **orthogonal**. If it is antisymmetric, the geometry is called **symplectic**. In the orthogonal case, $B(x, x)$ is written $Q(x)$. Actually, Q is the **quadratic form** associated to B, and they are related by

$$2B(x, y) = Q(x + y) - Q(x) - Q(y)$$

Actually, with the quadratic form we can redefine the inner and exterior product previously presented.

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First, define

Definition

For a $x = \{x_i\}_{i=1}^n \in \mathbb{R}^n$, we define its *reverse* as
$\check{x} = \{x_{n+1-i}\}_{i=1}^n = \{x_n, x_{n-1}, \cdots, x_1\}$

So,

$$A \cdot B := \frac{1}{2}[Q(A+B) - Q(A) - Q(B)]$$
$$A \wedge B := \frac{1}{2}[Q(A+\check{B}) - Q(A) - Q(B)]$$

For $\mathbb{R}^2$, with $Q = x_1^2 + x_2^2$ we can easily do

$$a \cdot b = \frac{1}{2}[Q(a+b) - Q(a) - Q(b)]$$
$$a \cdot b = \frac{1}{2}[(a_1 + b_1)^2 + (a_2 + b_2)^2 - (a_1^2 + a_2^2) - (b_1^2 + b_2^2)]$$
$$a \cdot b = \frac{1}{2}[2a_1 b_1 + 2a_2 b_2]$$
$$\Rightarrow a \cdot b = \frac{1}{2}[ab + ba]$$

$$a \wedge b = \frac{1}{2}[Q(a + \breve{b}) - Q(a) - Q(b)]$$
$$a \wedge b = \frac{1}{2}[(a_1 + b_2)^2 + (a_2 + b_1)^2 - (a_1^2 + a_2^2) - (b_1^2 + b_2^2)]$$
$$a \wedge b = \frac{1}{2}[2a_1 b_2 + 2a_2 b_1]$$
$$\Rightarrow a \cdot b = \frac{1}{2}[ab - ba]$$

and establish the Clifford Product again.

Evidently, Clifford Algebra could be obtained by means of the Tensor Algebra of the vector space in consideration.

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$$\mathcal{G}(V, Q) = T(V)/I$$

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$$\mathcal{G}(V, Q) = T(V)/I$$

Recalling that $x + y \in V$, it follows $(x + y) \otimes (x + y) - Q(x + y) \in I$

$$\begin{aligned}
&(x + y) \otimes (x + y) - Q(x + y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y - 2B - Q(x) - Q(y) \\
&= (x \otimes x - Q(x)) + (y \otimes y - Q(y)) + x \otimes y + y \otimes x - 2B \\
&= x \otimes y + y \otimes x - 2B + I_x + I_y \\
&= AB + BA - 2B
\end{aligned}$$

Where $\beta := Q(x + y) - Q(x) - Q(y)$ is the symmetric bilinear form induced by Q. Thus, we have

$$AB + BA = 2B \tag{5}$$

Is also remarkable that when Q(x)=0 we recover exterior algebra $\bigwedge V$.

$$\mathcal{C}(V,0) = \frac{\bigoplus_{k=1}^{\infty} V^{\otimes k}}{\{x \otimes x | x \in V\}} = \bigwedge V$$

And, here, we can simply write

$$a \wedge b = a \otimes b - b \otimes a$$

Exterior and Clifford Algebras

The relation between the Clifford Algebra and the Exterior Algebra is quite remarkable. For instance, $\mathcal{C}(V)$ carry a $\mathbb{Z}_2 -$ *graduation* (superalgebra) and the exterior algebra has a graduation in $\mathbb{Z}$. Remember that

Definition

An algebra A is **graded** if it is the direct sum of subspaces $A = A_1 \bigoplus A_2 \bigoplus \cdots$ s.t., $A_i A_j \subset A_{j+i}$, $\forall i, j \geq 0$. The elements of A_r are called of degree r.

Definition

A filtration of an algebra is a sequence of subspaces $A_0 \subset A_1 \subset \cdots$ s.t., $A_i A_j \subset A_{j+i}$, $\forall i, j \geq 0$.

We recognize the Clifford Algebra as a filtered algebra, meaning that we can find the associated graded algebra, being the Exterior Algebra, as we already shown. As well, a mapping from the graded algebra to the filtered algebra can be found. In the case of $\mathcal{C}$ and $\bigwedge V$ the map

$$\begin{aligned} q\colon \bigwedge V &\to \mathcal{C} \\ v &\mapsto q(v) \end{aligned}$$

defined by

$$q(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma_k} sgn(s) v_{s(1)} v_{s(2)} \cdots v_{s(k)} \tag{6}$$

Where σ_k is the group of permutations of $1, \cdots, k$ and $sgn(s) = \pm 1$ is the parity of the permutation s.

The relation between Clifford Algebra and Exterioir Algebra is a good example of what is called **deformation quantization**. This technique holds as well in between the Symmetric Algebra and the Weyl Algebra, another wonderful example. Deformation quantization is a formalization of the process of canonical quantization made in physics. When physicist are interested in quantize a theory, what they are doing is defining a map such that

$$q\colon S \to Op(\mathcal{H})$$

It maps smooth functions $f\colon M \to \mathbb{R}$ to operators $q(f)\colon \mathcal{H} \to \mathcal{H}$. We require q to satisfy:

1. $\mathbb{R}$ linearity: $q(rf+g) = rq(f) + q(g)$
2. Normalization $q(1) = \mathbf{1}$
3. Hermiticity $q(f)* = q(f)$
4. Dirac's quantum condition: $[q(f), q(g)] = -i\hbar q(f,g)$
5. Irreducibility condition: If $\{f_k\}_{k=1}^{n}$ is a complete set of observables, then $\{q(f_k)\}_{k=1}^{n}$ is a complete set of operators.

Quantization in mathematics

In the case of Classical Mechanics, you take the algebra of functions of the symplectic manifold and recover the algebra of operators in a Hilbert space. The product of the functions in the phase space (so called observables, $f \in C^\infty$) is defined pointwise, meaning that is commutative, while the algebra in the hilbert space is, actually, noncommutative. When quantizing a field, you are taking the correspondent commutative algebra of functions into a noncommutative one. Nevertheless, this algebra of functions is close related to the actual geometry underpinning it. We can extract geometric information from algebraic information, for example, two smooth manifolds are diffeomorphic if and only if the algebras of smooth real-valued functions on them are isomorphic; two locally compact Hausdorff spaces are homeomorphic if and only if their algebras of continuous real-valued functions that vanish at infinity are isomorphic. In this sense, a quantum mathematical object is geometrical fact in terms of their associated noncommutative algebra. For example, rouglhy speaking, a space is locally compact Hausdorff iff its algebra of continuous functions is commutative C*-algebra algebra. So a "quantum locally compact Hausdorff space" is the one with a non-commutative C*-algebra. So, physicaly, quantization acquires meaning and significance when the scale reaches the size of fundamental particles (even since nanoscales quantum corrections are relevant), but theoretically, the process in quantization has more to do with noncommutation.

Geometric Quantization and Noncommutative Geometry

On this efforts, the process of Geometric Quantization is a formalization of this notions which is under research and has succes taking the classical mechanics into de usual quantum mechanics. The process is quite sophisticated but can be achieved. Another great effort is that of a recent branch of mathematics called Noncommutative Geometry which try to do this exact thing of aproaching geometry through noncommutative algebras. Is remarkable to say that there exist a formalization of the Standard Model by means of this theory, called Noncommutative Standard Model, and actually can derive the lagrangian of the SM and even has calculations of the mass of the Higgs Boson.

Final Remarks

The process of quantization is well understood in symplectic manifolds, but is painfully obvius that there is a lot of research to do, for example, if the Riemannian structure could be quantized, we could be able to find a quantum version of the Einstein's Field Equations. Another thing is add to it the spin structure, which of curse the objects in the universe have, as the dirac equation has shown. So, find a novel object that supports Riemannian, Spin and Symplectic structures, as well as a process of quantization could be the next thing to do in physics.

References

[1]Clifford Algebras and Spinors. Second Edition. Pertti Lounesto, Helsinki Polytechnic Stadia.

[2]Orthogonal and Symplectic Clifford Algebras. Spinor Structures. Albert Crumeyrolle.

[3]Clifford Algebras as Filtered Algebras. Yaim Cooper. (http://www-math.mit.edu/ dav/cliffordfilt.pdf)

[4]Not Even Wrong. Notes on BRST VIII: Clifford Algebras. (https://www.math.columbia.edu/ woit/wordpress/?p=1387)

[5]Theo Johnson-Freyd (https://mathoverflow.net/users/78/theo-johnson-freyd), What is Quantization ?, URL (version: 2014-10-03): https://mathoverflow.net/q/6216